Noncommutativity and θ-locality

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Abstract
In this paper, we introduce the condition of θ-locality which can be used as
a substitute for microcausality in quantum field theory on noncommutative
spacetime. This condition is closely related to the asymptotic commutativity
which was previously used in nonlocal QFT. Heuristically, it means that
the commutators of observables behave at large spacelike separation like
$\exp(-|x - y|^2/\theta)$, where θ is the noncommutativity parameter. The rigorous
formulation given in the paper implies averaging fields with suitable test
functions. We define a test function space which most closely corresponds
to the Moyal $\star$-product and prove that this space is a topological algebra under
the star product. As an example, we consider the simplest normal ordered
monomial $: \phi \star \phi :$ and show that it obeys the θ-locality condition.

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1. Introduction

In this paper, we discuss the problem of formulating the causality principle in quantum
field theory on noncommutative spacetime. A noncommutative spacetime of d dimensions
is defined by replacing the coordinates x^μ of $\mathbb{R}^d$ by Hermitian operators $\hat{x}^\mu$ satisfying the
commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is a real antisymmetric $d \times d$ matrix, which will henceforth be assumed constant as in
most papers on this subject. The Weyl–Wigner correspondence between algebras of operators
and algebras of functions enables one to consider quantum field theories on noncommutative
spacetime as a form of nonlocal QFT described by an action, in which the ordinary product of

fields is replaced by the Moyal–Weyl–Groenewold star product

$$(f \star_\theta g)(x) = f(x) \exp\left(\frac{\mathrm{i}}{2} \overleftarrow{\partial_\mu} \theta^{\mu\nu} \overrightarrow{\partial_\nu}\right) g(x)$$

$$= f(x)g(x) + \sum_{n=1}^{\infty} \left(\frac{\mathrm{i}}{2}\right)^n \frac{1}{n!} \theta^{\mu_1\nu_1} \cdots \theta^{\mu_n\nu_n} \partial_{\mu_1} \cdots \partial_{\mu_n} f(x) \partial_{\nu_1} \cdots \partial_{\nu_n} g(x) \qquad (2)$$

(see, e.g., [1] for more details). Recent interest in noncommutative QFT was caused mainly by the fact that it occupies an intermediate position between the usual quantum field theory and string theory [2]. At present, considerable study is being given not only to actual models, but also to the conceptual framework of this theory. In particular, in [3–6], efforts were made to derive a corresponding generalization of the axiomatic approach [7–9]. Much attention is being given to the nonlocal effects inherent in noncommutative QFT. A comparison of the theories in which the time coordinate is involved in the noncommutativity and the theories with $\theta^{0\nu} = 0$ shows that these latter are preferable because they obey unitarity. In [3, 10, 11], it was argued, however, that in the case of space–space noncommutativity the usual causal structure with the light cone is replaced by a structure with a light wedge respecting the unbroken $SO(1,1) \times SO(2)$ symmetry. Since quantum fields are singular by their very nature, a comprehensive study of the question of causality must include finding an adequate space of test functions. In the standard formalism [7–9], quantum fields are taken to be tempered operator-valued distributions, which are defined on the Schwartz space $\mathscr{S}$ consisting of all infinitely differentiable functions of fast decrease. As noted in [3], the assumption of temperedness is open to the question in noncommutative QFT because of UV/IR mixing. Moreover, the correlation functions of some gauge-invariant operators admit an exponential growth at energies much larger than the noncommutativity scale [12], and this is an argument in favour of analytic test functions. The very structure of the star product (2), which is defined by an infinite-order differential operator, suggests that analytic test functions may be used in noncommutative QFT along with or instead of Schwartz's $\mathscr{S}$.

In [13], we argued that the appropriate test function spaces must be algebras under the Moyal $\star$-product and showed that the spaces S_α^β of Gelfand and Shilov [14] satisfy this condition if and only if $\alpha \geqslant \beta$. The space S_α^β consists of the smooth functions that decrease at infinity faster than exponentially with order $1/\alpha$ and a finite type, and whose Fourier transforms behave analogously but with order $1/\beta$. Clearly, all these spaces are contained in the space $\mathscr{S}$, which can be thought of as S_∞^∞. As shown in [13], the series (2) is absolutely convergent for any $f, g \in S_\alpha^\beta$ if and only if $\beta < 1/2$. However, the star multiplication has a unique continuous extension to any space S_α^β with $\alpha \geqslant \beta$. It is natural to use the spaces with $\beta < 1/2$ as an initial functional domain of quantum fields on noncommutative spacetime, but this does not rule out a possible extension to a larger test function space depending on the model under consideration. Recently, the use of spaces $S^\beta = S_\infty^\beta, \beta < 1/2$, was also advocated by Chaichian *et al* [15].

If $\beta < 1$, then the test functions are entire analytic, and the notion of support loses its meaning for the generalized functions that are defined on S_α^β or S^β and constitute their dual spaces $S_\alpha'^\beta$ and S'^β. Nevertheless, some basic theorems of the theory of distributions can be extended to these generalized functions because they retain the property of angular localizability [16, 17]. This property leads naturally to the condition of asymptotic commutativity, which was used in nonlocal QFT instead of local commutativity and was shown to ensure the existence of CPT-symmetry and the standard spin–statistics relation for nonlocal fields [18]. We already discussed in [19] how some of these proofs with test functions in S^0 can be adapted to noncommutative QFT. Here we intend to argue that quantum fields living on noncommutative spacetime indeed satisfy the asymptotic commutativity condition

and to explain the interrelation between this condition and the fundamental length scale which is determined by the noncommutativity parameter θ. To avoid notational clutter, we will use the one-index spaces of type S, although the two-index spaces provide a wider distributional framework.

In section 2, we introduce the test function space $\mathscr{S}^{1/2}$ which most closely corresponds to the Moyal star product. All spaces S^β with $\beta < 1/2$ are contained in this space, but it is smaller than $S^{1/2}$ and may be defined as a maximal space with the property that the series (2) is absolutely convergent for any pair of its elements. We also prove that $\mathscr{S}^{1/2}$ is a topological algebra under the star product. In section 3, two classes of spaces related to S^β, $\mathscr{S}^\beta$ and associated with cones in $\mathbb{R}^d$ are defined, and it is shown that these spaces are algebras under the $\star$-product for $\beta < 1/2$ in the former case and for $\beta \leqslant 1/2$ in the latter case. In section 4, the exact formulation of the asymptotic commutativity condition is given and its physical consequences are briefly outlined. In the same section, we introduce the notion of θ-locality. In section 5, we take, as a case in point, the normal ordered $\star$-square : $\phi \star \phi$: of the free scalar field ϕ and show that it obeys the conditions of asymptotic commutativity and θ-locality. Section 6 contains concluding remarks.

2. Test function spaces adequate to the Moyal star product

An advantage of the spaces S^β over $\mathscr{S}$ is their invariance under the action of infinite-order differential operators, the set of which increases with decreasing β. In what follows, we consider functions defined on $\mathbb{R}^d$ and use the usual multi-index notation:

$$\partial^\kappa = \frac{\partial^{|\kappa|}}{\partial x_1^{\kappa_1} \cdots \partial x_d^{\kappa_d}}, \qquad |\kappa| = \kappa_1 + \cdots + \kappa_d, \qquad \kappa^\kappa = \kappa_1^{\kappa_1} \cdots \kappa_d^{\kappa_d},$$

where $\kappa \in \mathbb{Z}_+^d$. Let $\beta \geqslant 0$, $B > 0$ and N be an integer. We denote by $S_N^{\beta,B}(\mathbb{R}^d)$ the Banach space of infinitely differentiable function with the norm

$$\|f\|_{B,N} = \sup_{x,\kappa}(1 + |x|)^N \frac{|\partial^\kappa f(x)|}{B^{|\kappa|}\kappa^{\beta\kappa}}. \tag{3}$$

We also write $S_N^{\beta,B}$ for this space when this cannot lead to confusion. Let us consider the operator

$$\sum_{\lambda \in \mathbb{Z}_+^d} c_\lambda \partial^\lambda \tag{4}$$

assuming that $\sum_\lambda c_\lambda z^\lambda$ has less than exponential growth of order $\leqslant 1/\beta$ and type b. In treatise [14], it was shown that under the condition $b < \beta/(\mathrm{e}^2 B^{1/\beta})$ the operator (4) maps the space $S_N^{\beta,B}$ to $S_N^{\beta,B'}$, where $B' = \mathrm{e}^\beta B$. This result can be improved by using the inequality $(k+l)^{k+l} \leqslant 2^{k+l}k^k l^l$. The assumption of order of growth, together with the Cauchy inequality, implies that $|c_\lambda| \leqslant C \prod_{j=1}^d r_j^{-\lambda_j} \mathrm{e}^{br_j^{1/\beta}}$ for any $r_j > 0$. Locating the minimum with respect to r_j, we obtain

$$|c_\lambda| \leqslant C \left(\frac{be}{\beta}\right)^{\beta|\lambda|} \frac{1}{\lambda^{\beta\lambda}}. \tag{5}$$

If $f \in S_N^{\beta,B}$, then we have

$$(1 + |x|)^N \left|\partial^\kappa \sum_\lambda c_\lambda \partial^\lambda f(x)\right| \leqslant \|f\|_{B,N} \sum_\lambda c_\lambda B^{|\kappa+\lambda|}(\kappa+\lambda)^{\beta(\kappa+\lambda)}$$

$$\leqslant \|f\|_{B,N} 2^{\beta|\kappa|} B^{|\kappa|}\kappa^{\beta\kappa} \sum_\lambda c_\lambda 2^{\beta|\lambda|} B^{|\lambda|}\lambda^{\beta\lambda}. \tag{6}$$

Suppose that

$$b < \frac{\beta}{2eB^{1/\beta}}, \tag{7}$$

then the last series in (6) converges by virtue of the inequality (5). Taking $B' \geqslant 2^\beta B$, we obtain $\left\| \sum_\lambda c_\lambda \partial^\lambda f \right\|_{B',N} \leqslant C' \|f\|_{B,N}$ and conclude that the operator (4) maps $S_N^{\beta,B}$ to $S_N^{\beta,B'}$ continuously.

Now we apply this consideration to the operator

$$\exp\left(\frac{\mathrm{i}}{2}\theta^{\mu\nu}\frac{\partial}{\partial x_1^\mu}\frac{\partial}{\partial x_2^\nu}\right) = \sum_{n=0}^\infty \left(\frac{\mathrm{i}}{2}\right)^n \frac{1}{n!}\theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n}\frac{\partial}{\partial x_1^{\mu_1}}\cdots\frac{\partial}{\partial x_1^{\mu_n}}\frac{\partial}{\partial x_2^{\nu_1}}\cdots\frac{\partial}{\partial x_2^{\nu_n}}. \tag{8}$$

Clearly, the order of the entire function $\exp\left((\mathrm{i}/2)\theta^{\mu\nu}Z_{1\mu}Z_{2\nu}\right)$ is equal to 2 and the type is less than or equal to $|\theta|/4$, where

$$|\theta| = \sum_{\mu<\nu}|\theta^{\mu\nu}|.$$

Hence we have the following theorem.

Theorem 1. *Let $B < 1/\sqrt{e|\theta|}$. Then the operator (8) maps the space $S_N^{1/2,B}(\mathbb{R}^{2d})$ continuously into the space $S_N^{1/2,B'}(\mathbb{R}^{2d})$, where $B' = B\sqrt{2}$. The series obtained by applying this operator to a function $f \in S_N^{1/2,B}(\mathbb{R}^{2d})$ is absolutely convergent in the norm $\|\cdot\|_{B',N}$.*

We define the countably-normed spaces $\mathscr{S}^\beta$ by

$$\mathscr{S}^\beta = \bigcap_{N,B} S_N^{\beta,B}. \tag{9}$$

A sequence f_n converges to $f \in \mathscr{S}^\beta$ if $\|f_n - f\|_{B,N} \to 0$ for every $B > 0$ and for every N. The foregoing leads directly to the following result.

Theorem 2. *The operator (8) maps the space $\mathscr{S}^{1/2}(\mathbb{R}^{2d})$ to itself continuously. Hence it is well defined and continuous on its dual space $\mathscr{S}'^{1/2}(\mathbb{R}^{2d})$. The series obtained by applying this operator to $f \in \mathscr{S}^{1/2}(\mathbb{R}^{2d})$ is absolutely convergent in each of the norms of $\mathscr{S}^{1/2}(\mathbb{R}^{2d})$.*

Below is given a description in terms of the Fourier transform, which shows that the operator (8) is bijective on $\mathscr{S}^{1/2}$ and so it is a linear topological isomorphism of $\mathscr{S}^{1/2}$ as well as of $\mathscr{S}'^{1/2}$. Analogous statements certainly hold for any $\mathscr{S}^\beta$ with $\beta \leqslant 1/2$, but $\mathscr{S}^{1/2}$ is the largest of these spaces and most closely corresponds to the operator (8) and hence to the Moyal product (2). In what follows, we use the notation

$$\partial_{x_1}\theta\partial_{x_2} = \theta^{\mu\nu}\frac{\partial}{\partial x_1^\mu}\frac{\partial}{\partial x_2^\nu}.$$

The map $\mathscr{S}^\beta(\mathbb{R}^d) \times \mathscr{S}^\beta(\mathbb{R}^d) \to \mathscr{S}^\beta(\mathbb{R}^d)$ that takes each pair (f,g) to the function $f \star g$ can be considered as the composite map

$$\mathscr{S}^\beta(\mathbb{R}^d) \times \mathscr{S}^\beta(\mathbb{R}^d) \xrightarrow{\otimes} \mathscr{S}^\beta(\mathbb{R}^{2d}) \xrightarrow{\mathrm{e}^{(\mathrm{i}/2)\partial_{x_1}\theta\partial_{x_2}}} \mathscr{S}^\beta(\mathbb{R}^{2d}) \xrightarrow{\widehat{\mathrm{m}}} \mathscr{S}^\beta(\mathbb{R}^d), \tag{10}$$

where the first arrow takes (f,g) to the function $(f \otimes g)(x_1,x_2) = f(x_1)g(x_2)$, the second arrow is the action of operator (8) and the third arrow is the restriction of elements of $\mathscr{S}^\beta(\mathbb{R}^{2d})$ to the diagonal $x_1 = x_2$. The first map is obviously continuous, and we now argue that the third map is also continuous. Although the spaces $\mathscr{S}^\beta$ are not invariant under the Fourier transformation, they closely resemble the Schwartz space $\mathscr{S}$ in their other properties. These

spaces are complete and metrizable, i.e., belong to the class of Fréchet spaces. Furthermore, they are Montel spaces (or perfect in nomenclature of [14]) and nuclear. An analogue of Schwartz's kernel theorem states that $\mathscr{S}^{\beta}(\mathbb{R}^{2d})$ coincides with the completion of the tensor product $\mathscr{S}^{\beta}(\mathbb{R}^{d}) \otimes_{\pi} \mathscr{S}^{\beta}(\mathbb{R}^{d})$ equipped with the projective topology. Therefore, the set of continuous bilinear maps $\mathscr{S}^{\beta}(\mathbb{R}^{d}) \times \mathscr{S}^{\beta}(\mathbb{R}^{d}) \to \mathscr{S}^{\beta}(\mathbb{R}^{d})$ can be identified with the set of continuous linear maps $\mathscr{S}^{\beta}(\mathbb{R}^{2d}) \to \mathscr{S}^{\beta}(\mathbb{R}^{d})$. In particular, the linear map $\widehat{\mathsf{m}}$ corresponds to the ordinary pointwise multiplication $\mathsf{m}\colon (f, g) \to f \cdot g$, and its continuity follows from (and amounts to) the fact that $\mathscr{S}^{\beta}(\mathbb{R}^{d})$ is a topological algebra under the ordinary multiplication. We thus get the following theorem.

Theorem 3. *The spaces $\mathscr{S}^{\beta}(\mathbb{R}^{d})$ with $\beta \leqslant 1/2$ are topological algebras under the Moyal $\star$-product. If $f, g \in \mathscr{S}^{\beta}(\mathbb{R}^{d})$, where $\beta \leqslant 1/2$, then the series (2) is absolutely convergent in this space.*

Another way of proving this is to estimate the expression $(1 + |x|)^{N} |\partial^{\kappa}(f \star g)(x)|$ with the use of Leibniz's formula. Such a computation is almost identical to the proof of theorem 4 in paper [13] dealing with the spaces S_{α}^{β}.

The Gelfand–Shilov spaces S^{β} are constructible from the spaces $S_{N}^{\beta, B}$ in the following way:

$$S^{\beta} = \bigcup_{B>0} S^{\beta, B}, \qquad S^{\beta, B} = \bigcap_{B'>B, N \in \mathbb{Z}_{+}} S_{N}^{\beta, B'}. \tag{11}$$

A sequence f_{n} is said to be convergent to an element $f \in S^{\beta}$ if there is a $B > 0$ such that all f_{n} and f are contained in the space $S^{\beta, B}$ and $f_{n} \to f$ in each of its norms.

Gelfand and Shilov [14] have shown that the spaces S^{β} are algebras under the pointwise multiplication and that this operation is separately continuous in their topology. Mityagin [20] has proved that these spaces are nuclear. Another proof is given in [21], where in addition their completeness is established and the corresponding kernel theorem is proved. From this theorem, it follows that the set of separately continuous bilinear maps $S^{\beta}(\mathbb{R}^{d}) \times S^{\beta}(\mathbb{R}^{d}) \to S^{\beta}(\mathbb{R}^{d})$ is identified with the set of continuous linear maps $S^{\beta}(\mathbb{R}^{2d}) \to S^{\beta}(\mathbb{R}^{d})$. We combine these facts in a manner analogous to that used in the case of $\mathscr{S}^{\beta}$ and suppose that β satisfies the strict inequality $\beta < 1/2$. Then $\mathrm{e}^{b|z|^{1/2}} \leqslant C_{\epsilon} \mathrm{e}^{\epsilon |z|^{1/\beta}}$, where $\epsilon > 0$ can be taken arbitrarily small, and we obtain the following result.

Theorem 4. *The operator (8) maps every space $S^{\beta}(\mathbb{R}^{2d})$ with $\beta < 1/2$ to itself continuously. The spaces $S^{\beta}(\mathbb{R}^{d})$, where $\beta < 1/2$, are algebras under the Moyal $\star$-product, and the star multiplication is separately continuous under their topology. If $f, g \in S^{\beta}(\mathbb{R}^{d})$, then the series (2) converges absolutely in this space.*

The Fourier transformation $\mathcal{F}\colon f(x) \to \hat{f}(p) = \int f(x)\, \mathrm{e}^{\mathrm{i} p \cdot x}\, \mathrm{d}x$ converts S^{β} to the space S_{β} which consists of all smooth functions satisfying the inequalities

$$|\partial^{\kappa} h(p)| \leqslant C_{\kappa}\, \mathrm{e}^{-|p/B|^{1/\beta}}, \qquad \text{for some } B(h) > 0 \text{ and for every } \kappa,$$

whereas $\mathscr{S}_{\beta} = \mathcal{F}[\mathscr{S}^{\beta}]$ consists of the functions satisfying

$$|\partial^{\kappa} h(p)| \leqslant C_{\kappa, B}\, \mathrm{e}^{-|p/B|^{1/\beta}}, \qquad \text{for each } B > 0 \text{ and for every } \kappa.$$

The operator (8) turns into the multiplication of the Fourier transforms by the function

$$\mathrm{e}^{-(\mathrm{i}/2) p_{1} \theta p_{2}}, \qquad \text{where} \quad p_{1} \theta p_{2} \stackrel{\text{def}}{=} p_{1\mu} \theta^{\mu\nu} p_{2\nu}. \tag{12}$$

Clearly, this function is a multiplier of $\mathscr{S}_{\beta}(\mathbb{R}^{2d})$ and of $S_{\beta}(\mathbb{R}^{2d})$ for any β. Hence the operator (8) admits continuous extension to all these spaces, and this extension is unique because

$S_0 = C_0^\infty$ is dense in each of them. It follows that the $\star$-product also has a unique continuous extension to the spaces with $\beta > 1/2$. This extension is defined by

$$(f \times g)(x) = \frac{1}{(2\pi)^{2d}} \iint \hat{f}(q)\hat{g}(p)\, e^{-iqx-ipx-(i/2)q\theta p}\, dq\, dp$$

and is often called the 'twisted product' [22]. The uniqueness of the extension entitles us to identify the operation $(f, g) \to f \times g$ with the $\star$-multiplication and to say that $\mathscr{S}^\beta$ and S^β are star product algebras for any β. However, proposition 2 in [13] shows that every space $\mathscr{S}^\beta$ with $\beta > 1/2$ and every S^β with $\beta \geqslant 1/2$ contain functions for which the series (2) is not convergent in the topology of these spaces.

3. Test function algebras over cones in $\mathbb{R}^d$

The operator (8) is nonlocal, but when acting on the functionals defined on $\mathscr{S}^\beta$ or on S^β, it preserves the property of a rapid decrease along a given direction of $\mathbb{R}^{2d}$ if a functional has such a property. In order for this statement to be given a precise mathematical meaning, we use spaces which are related to $\mathscr{S}^\beta(\mathbb{R}^d)$ and $S^\beta(\mathbb{R}^d)$, but associated with cones in $\mathbb{R}^d$. Such sheaves of spaces arise naturally in nonlocal quantum field theory, see [16, 17].

Let U be an open connected cone in $\mathbb{R}^d$. We denote by $S_N^{\beta,B}(U)$ the space of all infinitely differentiable functions on U with the finite norm

$$\|f\|_{U,B,N} = \sup_{x\in U;\kappa} (1 + |x|)^N \frac{|\partial^\kappa f(x)|}{B^{|\kappa|}\kappa^{\beta\kappa}}. \tag{13}$$

The spaces $\mathscr{S}^\beta(U)$, $S^{\beta,B}(U)$ and $S^\beta(U)$ are constructed from $S_N^{\beta,B}(U)$ by formulae analogous to (9) and (11). If $\beta \leqslant 1$, then all elements of these spaces can be continued analytically to the whole of $\mathbb{C}^d$ and this definition can be rewritten in terms of complex variables. Using the Taylor and Cauchy formulae, it is easy to verify (see, e.g., [16] for details) that the space $S^\beta(U)$ with $\beta < 1$ coincides with the space of all entire analytic functions satisfying the inequalities

$$|f(z)| \leqslant C_N(1 + |x|)^{-N} e^{d(Bx,U)^{1/(1-\beta)}+|By|^{1/(1-\beta)}}, \qquad N = 0, 1, \ldots, \tag{14}$$

where $z = x + iy$, $d(x, U) = \inf_{\xi\in U}|x - \xi|$ is the distance from x to U and the constants C_N, B depend on f. This space is independent of the choice of the norm $|\cdot|$ on $\mathbb{R}^d$, because all these norms are equivalent. We also note that $d(Bx, U) = Bd(x, U)$ since U is a cone. The analytic continuations of the elements of $\mathscr{S}^\beta(U)$ satisfy analogous inequalities for every B and for every N with constants $C_{B,N}$ instead of C_N. This representation makes it clear that the spaces $S^\beta(U)$ and $\mathscr{S}^\beta(U)$, where $\beta < 1$, are algebras under the ordinary multiplication.

The arguments used in the proofs of theorems 1 and 2 are completely applicable to the spaces over cones and furnish the following result.

Theorem 5. *Let U be an open cone in $\mathbb{R}^{2d}$. If $B < 1/\sqrt{e|\theta|}$, then the operator (8) maps the normed space $S_N^{1/2,B}(U)$ to $S_N^{1/2,B'}(U)$, where $B' = B\sqrt{2}$, and is bounded. Each of the spaces $\mathscr{S}^\beta(U)$ with $\beta \leqslant 1/2$ and $S^\beta(U)$ with $\beta < 1/2$ is continuously mapped by this operator into itself. Consequently, it is defined and continuous on their dual spaces $\mathscr{S}'^\beta(U)$, $S'^\beta(U)$. The series obtained by applying the operator (8) to the elements of these spaces are absolutely convergent.*

As is shown in [21], the spaces $S^{\beta,B}(U)$ are nuclear. It immediately follows that $\mathscr{S}^\beta(U)$ and $S^\beta(U)$ also have this property. Theorem 6 of [21] states that the space $S^\beta(U \times U)$ coincides with the completion of the tensor product $S^\beta(U) \otimes_\iota S^\beta(U)$ endowed with the

inductive topology. Let $f, g \in S^\beta(U)$. In complete analogy to the reasoning of section 2, we can decompose the map $(f, g) \to f \star g$ as follows:

$$S^\beta(U) \times S^\beta(U) \xrightarrow{\otimes} S^\beta(U \times U) \xrightarrow{e^{(i/2)\partial_{x_1}\theta\partial_{x_2}}} S^\beta(U \times U) \xrightarrow{\widehat{m}} S^\beta(U). \tag{15}$$

The former map in (15) is separately continuous and the latter two are continuous. (As before, we denote by $\widehat{m}$ the linear map that canonically corresponds to the ordinary product.) A similar representation certainly holds for the spaces $\mathscr{S}^\beta(U)$, and in that case we have even a simpler situation because these are Fréchet spaces and we need not distinguish between separately continuous and continuous linear maps. We thus get the following theorem.

Theorem 6. *Let U be an open cone in $\mathbb{R}^d$. Every space $\mathscr{S}^\beta(U)$ with $\beta \leqslant 1/2$ is a topological algebra under the Moyal $\star$-product. If $\beta < 1/2$, then $S^\beta(U)$ also is an algebra with respect to the Moyal product and the $\star$-multiplication is separately continuous under its topology. The series (2) converges absolutely in these spaces for each pair of their elements.*

The special convenience of S^β (and S_α^β) is that the generalized functions defined on these spaces of analytic test functions have been shown to possess the property of angular localizability, which is specified in the following manner. We say that a functional $v \in S'^\beta(\mathbb{R}^d)$ is carried by a closed cone $K \subset \mathbb{R}^d$ if v admits a continuous extension to every space $S^\beta(U)$, where $U \supset K\backslash\{0\}$. This property is equivalent to the existence of a continuous extension to the space

$$S^\beta(K) = \bigcup_{U \supset K\backslash\{0\}} S^\beta(U) \tag{16}$$

endowed with the topology induced by the family of injections $S^\beta(U) \to S^\beta(K)$. When such an extension exists, it is unique because S^β is dense in $S^\beta(U)$ and in $S^\beta(K)$ by theorem 5 of [23]. The representation (14) makes it clear that outside K elements of $S^\beta(K)$ can have an exponential growth of order $1/(1 - \beta)$ and a finite type. Hence we are entitled to interpret the existence of a nontrivial carrier cone of $v \in S^\beta(\mathbb{R}^d)$ as a falloff property of this functional in the complementary cone or more specifically as a decrease faster than exponentially with order $1/(1 - \beta)$ and maximum type. Moreover, the relation

$$S'^\beta(K_1 \cap K_2) = S'^\beta(K_1) \cap S'^\beta(K_2) \tag{17}$$

holds, which implies that every element of $S'^\beta(\mathbb{R}^d)$ has a unique minimal closed carrier cone in $\mathbb{R}^d$. This fact has been established in [16, 17] for $0 < \beta < 1$, and a detailed proof of the relation (17) for the most complicated case $\beta = 0$ is available in [21]. Clearly, the spaces $S^\beta(K)$ over closed cones also are algebras under the Moyal $\star$-product if $\beta < 1/2$.

4. Asymptotic commutativity and θ-locality

It is believed that a mathematically rigorous theory of quantum fields on noncommutative spacetime shall adopt the basic assumption of the axiomatic approach [7–9] that quantum fields are operator-valued generalized functions with a common, dense and invariant domain D in the Hilbert space of states. The optimal test function spaces may be model-dependent, but the above consideration shows that in any case the space $\mathscr{S}^{1/2}$ and the spaces S^β with $\beta < 1/2$, as well as their related spaces over cones, are attractive for use in noncommutative QFT.

Analytic test functions were used in nonlocal field theory over many years and it would be reasonable to draw this experience. The axiomatic formulation of nonlocal QFT developed in [16–18] is based on the idea of changing local commutativity to an asymptotic commutativity

condition, which means that the commutator or anticommutator of any two fields of the theory is carried by the cone

$$\overline{\mathbb{V}} \times \mathbb{R}^d = \{(x, x') \in \mathbb{R}^{2d} : (x - x')^2 \geqslant 0\}. \tag{18}$$

In more exact terms, if the fields ϕ, ψ are defined on the test function space $S^\beta(\mathbb{R}^d)$, $\beta < 1$, then either

$$\langle \Phi [\phi(x), \psi(x')]_- \Psi \rangle \tag{19}$$

or

$$\langle \Phi [\phi(x), \psi(x')]_+ \Psi \rangle \tag{20}$$

is carried by the cone (18) for all $\Phi, \Psi \in D$. The matrix elements (19), (20) can be regarded as generalized functions on $\mathbb{R}^d$ because S^β is nuclear and the relation $S^\beta(\mathbb{R}^d) \hat{\otimes}_\iota S^\beta(\mathbb{R}^d) = S^\beta(\mathbb{R}^{2d})$ holds. The asymptotic commutativity condition becomes weaker with decreasing β. For $\beta = 0$, it means that the commutator of observable fields averaged with test functions in S^0 decreases at spacelike separation no worse than exponentially with order 1 and maximum type. Together with other Wightman axioms, this condition ensures the existence of the CPT-symmetry operator and the normal spin–statistics relation for the nonlocal quantum fields. The proofs [18] of these theorems use the notion of the analytic wave front set of distributions in an essential way. This generalization of the local commutativity axiom also preserves the cluster decomposition property of the vacuum expectation values. As shown in [24], it preserves even the strong exponential version of this property if the theory has a mass gap. This makes possible interpreting the nonlocal QFT subject to the asymptotic commutativity condition in terms of the particle scattering because the cluster decomposition property plays a key role in constructing the asymptotic states and the S-matrix.

In [19], we discussed some peculiarities of using the analytic test functions in quantum field theory on noncommutative spacetime for the case of a charged scalar field and space–space noncommutativity. We have shown that this theory has CPT-symmetry if it satisfies a suitably modified condition of asymptotic commutativity. This modification uses the generalization [21, 25] of the notion of carrier cone to the bilinear forms.

The test function spaces S^β, $\beta < 1/2$, are convenient for use in quantum field theory on noncommutative spacetime because they are algebras under the $\star$-product and the generalized functions defined on them have the property of angular localizability, which enables one to apply analogues of some basic theorems of Schwartz's theory of distributions. Moreover, $S^\beta(\mathbb{R}^d)$ are invariant under the affine transformations of coordinates and the spaces of this kind over the light cone are invariant under the Poincaré group. The asymptotic commutativity provides a way of formulating causality in noncommutative QFT, but it is insensitive to the magnitude of the noncommutativity parameter which determines the fundamental length scale. The above analysis suggests that a more accurate formulation can be obtained by using spaces $S^{1/2, B}$. The nonlocal effects in quantum field theory on noncommutative spacetime are determined by the structure of the Moyal $\star$-product, and one might expect that in this theory each of the matrix elements (19) (or (20) for unobservable fields) admits a continuous extension to the space

$$S^{1/2, B}(\overline{\mathbb{V}} \times \mathbb{R}^d), \qquad \text{where} \quad B \sim \frac{1}{\sqrt{|\theta|}}. \tag{21}$$

(In general, B may depend on the fields ϕ, ψ and the states Φ, Ψ.) This condition will be called θ-locality. Clearly, it is stronger than the asymptotic commutativity condition stated for $\beta < 1/2$, but it is also consistent with the Poincaré covariance. Conceivably, the θ-locality expresses the absence of acausal effects on scales much larger than the fundamental scale

$\Lambda \sim \sqrt{|\theta|}$. If such is the case, this assumption might be called macrocausality. It should be emphasized that we do not assume here that the fields are defined only on the analytic test functions. It is quite possible that their matrix elements are usual tempered distributions. In other words, we use $S^{1/2,B}$ as a tool for formulating causality rather than as the functional domain of definition of fields. In the next section, we reconsider from this standpoint a typical example which was used in [26, 27] for showing the violation of microcausality in quantum field theory on noncommutative spacetime.

5. An example

Let ϕ be a free neutral scalar field of mass m in a spacetime of d dimensions and let

$$\mathcal{O}(x) \equiv\; : \phi \star \phi : (x) = \lim_{x_1, x_2 \to x} : \phi(x_1)\phi(x_2) :$$

$$+ \sum_{n=1}^{\infty} \left(\frac{\mathrm{i}}{2} \right)^n \frac{1}{n!} \theta^{\mu_1 \nu_1} \cdots \theta^{\mu_n \nu_n} \lim_{x_1, x_2 \to x} : \partial_{\mu_1} \cdots \partial_{\mu_n} \phi(x_1) \partial_{\nu_1} \cdots \partial_{\nu_n} \phi(x_2) :. \quad (22)$$

Every term in (22) is well defined as a Wick binomial. Chaichian *et al* [26] and Greenberg [27] studied the question of microcausality in noncommutative QFT for the choice $\mathcal{O}$ as a sample observable. Specifically, they considered the matrix element

$$\langle 0|[\mathcal{O}(x), \mathcal{O}(y)]_-|p_1, p_2\rangle$$

at $x^0 = y^0$. In the case of space–space noncommutativity, with $\theta^{12} = -\theta^{21} \neq 0$ and the other elements of the θ-matrix equal to zero, the commutator $[\mathcal{O}(x), \mathcal{O}(y)]_-$ vanishes in the light wedge $(x^0 - y^0)^2 < (x^3 - y^3)^2$, but Greenberg found that $[\mathcal{O}(x), \partial_\nu \mathcal{O}(y)]_-$ fails to vanish outside this wedge and so violates microcausality. We shall show that nevertheless the θ-locality condition is fulfilled for this observable and this result holds irrespectively of the type of noncommutativity.

First, we calculate the vacuum expectation value

$$\mathcal{W}(x, y; z_1, z_2) = \langle 0|\mathcal{O}(x)\mathcal{O}(y) : \phi(z_1)\phi(z_2) : |0\rangle. \quad (23)$$

We use the Wick theorem and express

$$\langle 0| : \phi(x_1)\phi(x_2) :: \phi(y_1)\phi(y_2) :: \phi(z_1)\phi(z_2) : |0\rangle$$

in terms of the two-point function

$$w(x - y) = \langle 0|\phi(x)\phi(y)|0\rangle = \frac{1}{(2\pi)^{d-1}} \int \mathrm{e}^{-\mathrm{i}k \cdot (x-y)} \vartheta(k^0) \delta(k^2 - m^2)\,\mathrm{d}k.$$

Applying then the relation

$$\lim_{x_1, x_2 \to x} \exp\left(\frac{\mathrm{i}}{2} \partial_{x_1} \theta \partial_{x_2} \right) \mathrm{e}^{\mathrm{i}k \cdot x_1} \mathrm{e}^{\mathrm{i}p \cdot x_2} \equiv \mathrm{e}^{\mathrm{i}k \cdot x} \star \mathrm{e}^{\mathrm{i}p \cdot x} = \mathrm{e}^{-(\mathrm{i}/2)k\theta p} \mathrm{e}^{\mathrm{i}(k+p) \cdot x},$$

we obtain

$$\mathcal{W}(x, y; z_1, z_2) = 4 \int \frac{\mathrm{d}k\,\mathrm{d}p_1\,\mathrm{d}p_2}{(2\pi)^{3(d-1)}} \vartheta(k^0) \delta(k^2 - m^2) \prod_{i=1}^{2} \vartheta(p_i^0) \delta(p_i^2 - m^2)$$

$$\times \cos\left(\frac{1}{2} k\theta p_i \right) \mathrm{e}^{-\mathrm{i}k \cdot (x-y) - \mathrm{i}p_1 \cdot (x-z_1) - \mathrm{i}p_2 \cdot (y-z_2)} + (z_1 \leftrightarrow z_2). \quad (24)$$

This formal derivation should be accompanied by a comment. The function

$$\cos\left(\frac{1}{2} k\theta p_1 \right) \cos\left(\frac{1}{2} k\theta p_2 \right)$$

is a multiplier for the Schwartz space $\mathscr{S}$, and hence the right-hand side of (24) is well defined as a tempered distribution. This distribution is obtained by applying the operator

$$\cos\left(\tfrac{1}{2}\partial_x\theta\partial_{z_1}\right)\cos\left(\tfrac{1}{2}\partial_y\theta\partial_{z_2}\right) \tag{25}$$

to the distribution

$$4\int \frac{\mathrm{d}k\,\mathrm{d}p_1\,\mathrm{d}p_2}{(2\pi)^{3(d-1)}}\vartheta(k^0)\delta(k^2-m^2)\prod_{i=1}^{2}\vartheta\left(p_i^0\right)\delta\left(p_i^2-m^2\right)\mathrm{e}^{-\mathrm{i}k\cdot(x-y)-\mathrm{i}p_1\cdot(x-z_1)-\mathrm{i}p_2\cdot(y-z_2)}$$
$$+\,(z_1\leftrightarrow z_2).$$

By theorem 2, the operator (25) is defined and is continuous on $\mathscr{S}^{1/2}(\mathbb{R}^{4d})$ (and on any space $S^{\beta}(\mathbb{R}^{4d})$ with $\beta < 1/2$) and the power series expansion of (24) in θ is weakly convergent to $\mathscr{W}$ in the dual space $\mathscr{S}'^{1/2}$. This implies the strong convergence because $\mathscr{S}^{1/2}$ is a Montel space. However, that is not to say that this expansion converges to $\mathscr{W}$ in the topology of the space $\mathscr{S}'$ of tempered distributions.

Using (24), we obtain

$$\langle 0|[\mathcal{O}(x),\mathcal{O}(y)]_- : \phi(z_1)\phi(z_2)\colon |0\rangle$$
$$= 4\int \frac{\mathrm{d}k\,\mathrm{d}p_1\,\mathrm{d}p_2}{(2\pi)^{3(d-1)}}\epsilon(k^0)\delta(k^2-m^2)\prod_{i=1}^{2}\mathrm{d}\theta\left(p_i^0\right)\delta\left(p_i^2-m^2\right)$$
$$\times \cos\left(\frac{1}{2}k\theta p_i\right)\mathrm{e}^{-\mathrm{i}k\cdot(x-y)-\mathrm{i}p_1\cdot(x-z_1)-\mathrm{i}p_2\cdot(y-z_2)} + (z_1\leftrightarrow z_2), \tag{26}$$

which agrees with formula (7) of [27].

Theorem 7. *The restriction of the distribution (26) to $S^{1/2}$ has a continuous extension to the space $S^{1/2,B}(\mathbb{V}\times\mathbb{R}^{3d})$, where $B < 1/\sqrt{e|\theta|}$ and*

$$\mathbb{V}\times\mathbb{R}^{3d} = \{(x,y,z_1,z_2)\in\mathbb{R}^{4d}\colon (x-y)^2 > 0\}. \tag{27}$$

A fortiori, the restriction of this distribution to any space $S^{\beta}(\mathbb{R}^{4d})$ with $\beta < 1/2$ is strongly carried by the closed cone $\overline{\mathbb{V}}\times\mathbb{R}^{3d}$.

Proof. Let $B' = B\sqrt{2}$. The restriction of (26) to $S^{1/2,B'}(\mathbb{R}^{4d})$ is obtained by applying the operator (25) to the restriction of

$$D(x,y;z_1,z_2) \equiv \langle 0|[:\phi^2:(x),:\phi^2:(y)]_- :\phi(z_1)\phi(z_2):|0\rangle$$
$$= 4\mathrm{i}\Delta(x-y)w(x-z_1)w(y-z_2) + (z_1\leftrightarrow z_2). \tag{28}$$

Clearly, $D(x,y;z_1,z_2)$ vanishes for $(x-y)^2 < 0$ and the restriction $D|S^{1/2,B'}(\mathbb{R}^{4d})$ has a continuous extension $\widetilde{D}$ to the space $S^{1/2,B'}(\mathbb{V}\times\mathbb{R}^{3d})$. This extension can be defined by $(\widetilde{D},f) = (D,\chi f)$, where χ is a multiplier of the Schwartz space, which is equal to 1 on an ϵ-neighbourhood of $\overline{\mathbb{V}}\times\mathbb{R}^{3d}$ and to zero outside the 2ϵ-neighbourhood. Such a multiplier satisfies the uniform estimate $|\partial^{\kappa}\chi| \leqslant C_{\kappa}$, and the multiplication by χ maps $S^{1/2,B'}(\mathbb{V}\times\mathbb{R}^{3d})$ into $\mathscr{S}(\mathbb{R}^{4d})$ continuously. By theorem 5, applying (25) to $\widetilde{D}$, we obtain a continuous extension of the functional (26) to the space $S^{1/2,B}(\mathbb{V}\times\mathbb{R}^{3d})$. This proves theorem 7. We point out once again that this theorem holds for any matrix θ and in particular for both space–space and spacetime noncommutativity. $\qquad\square$

6. Concluding remarks

Our analysis shows that the θ-locality condition or the weaker condition of asymptotic commutativity for the restrictions of fields to the test function spaces $S^\beta, \beta < 1/2$, can serve as a substitute of microcausality in quantum field theory on noncommutative spacetime even though the fields are tempered. The character of singularity is certainly dependent on the model, but multiplication by the exponential (12) alone cannot spoil temperedness. As stressed in [6, 28], any attempt to replace microcausality by a weaker requirement must take the theorem on the global nature of local commutativity into consideration. The Borchers and Pohlmeyer version [29] of this theorem states that local commutativity follows from an apparently weaker assumption that $[\phi(x), \psi(x')]_\pm$ decreases at large spacelike separation faster than exponentially of order 1. The example : $\phi \star \phi$: discussed above demonstrates that this theorem is inapplicable to the asymptotic commutativity condition and that this condition does not imply local commutativity. The point is that the fast decrease at spacelike separation is understood here differently than in [29], as a property of the field (anti)commutators averaged with appropriate test functions. We have restricted our consideration to the specific matrix element of the commutator, but the technique developed in [30] enables one to construct the operator realization of : $\phi \star \phi$: in the state space of ϕ and to prove that in this instance the θ-locality condition is completely fulfilled. In combination with the usual relativistic transformation law of states and fields, the asymptotic commutativity ensures the existence of CPT-symmetry and the normal spin–statistics relation for nonlocal fields [18]. One might expect that in noncommutative QFT similar conclusions can be deduced from a suitable combination of the θ-locality and the twisted Poincaré covariance [6, 31] which has currently received much attention.

Most, if not all, of the results established above for S^β can readily be extended to the spaces S^β_α whose topological structure is even simpler. In particular, a theorem similar to theorem 1 holds with $S^{1/2,B}_{\alpha,A}$ in place of $S^{1/2,B}_N$. Analogues of theorems 2 and 3 hold for $\mathscr{S}^\beta_\alpha = \bigcap_{A,B} S^{\beta,B}_{\alpha,A}$, where $\beta \leqslant 1/2$ and $\alpha > 1 - \beta$. An analogue of theorem 4 is valid for $S^\beta_\alpha = \bigcup_{A,B} S^{\beta,B}_{\alpha,A}$ with $\beta < 1/2$ and $\alpha \geqslant 1 - \beta$. Of course, analogues of theorems 5 and 6 hold with the same replacements.

References

[1] Szabo R J 2003 Quantum field theory on noncommutative spaces *Phys. Rep.* **378** 207–99 (*Preprint* hep-th/0109162)
[2] Seiberg N and Witten E 1999 String theory and noncommutative geometry *J. High Energy Phys.* JHEP09(1999)032 (*Preprint* hep-th/9908142)
[3] Alvarez-Gaume L and Vazquez-Mozo M A 2003 General properties of noncommutative field theories *Nucl. Phys.* B **668** 293–321 (*Preprint* hep-th/0305093)
[4] Chaichian M, Mnatsakanova M N, Nishijima K, Tureanu A and Vernov Yu A 2004 Towards an axiomatic formulation of noncommutative field theories *Preprint* hep-th/0402212
[5] Franco D H T and Polito C M M 2005 A new derivation of the CPT and spin–statistics theorems in noncommutative field theories *J. Math. Phys.* **46** 083503 (*Preprint* hep-th/0403028)

[6] Fiore G and Wess J 2007 On 'full' twisted Poincaré symmetry and QFT on Moyal–Weyl spaces *Phys. Rev.* D **75** 105022 (*Preprint* hep-th/0701078)

[7] Streater R F and Wightman A S 1964 *PCT, Spin and Statistics and All That* (New York: Benjamin)

[8] Jost R 1965 *The General Theory of Quantized Fields* (Providence, RI: American Mathematical Society)

[9] Bogoliubov N N, Logunov A A, Oksak A I and Todorov I T 1990 *General Principles of Quantum Field Theory* (Dordrecht: Kluwer)

[10] Liao Yi and Sibold K 2002 Spectral representation and dispersion relations in field theory on noncommutative space *Phys. Lett.* B **549** 352 (*Preprint* hep-th/0209221)

[11] Chu C-S, Furuta K and Inami T 2006 Locality, causality and noncommutative geometry *Int. J. Mod. Phys.* A **21** 67 (*Preprint* hep-th/0502012)

[12] Gross D J, Hashimoto A and Itzhaki N 2000 Observables of non-commutative gauge theories *Adv. Theor. Math. Phys.* **4** 893–928 (*Preprint* hep-th/0008075)

[13] Soloviev M A 2007 Star product algebras of test functions *Theor. Math. Phys.* **153** 1351–63 (*Preprint* 0708.0811)

[14] Gelfand I M and Shilov G E 1968 *Generalized Functions* vol 2 (New York: Academic)

[15] Chaichian M, Mnatsakanova M N, Tureanu A and Vernov Yu A 2007 Test function space in noncommutative quantum field theory *Preprint* 0706.1712

[16] Fainberg V Ya and Soloviev M A 1992 Nonlocalizability and asymptotical commutativity *Theor. Math. Phys.* **93** 1438 (*Preprint* hep-th/9211099)

[17] Soloviev M A 1993 Beyond the theory of hyperfunctions *Developments in Mathematics: The Moscow School* ed V Arnold and M Monastyrsky (London: Chapman and Hall) pp 131–93

[18] Soloviev M A 1999 PCT, spin and statistics and analytic wave front set *Theor. Math. Phys.* **121** 1377–96 (*Preprint* hep-th/0605243)

[19] Soloviev M A 2006 Axiomatic formulations of nonlocal and noncommutative field theories *Theor. Math. Phys.* **147** 660–9 (*Preprint* hep-th/0605249)

[20] Mityagin B S 1960 Nuclearity and other properties of the spaces of type S *Russ. Math. Surv.* **9** 317–28 (in Russian)

[21] Soloviev M A 2006 Decomposition theorems and kernel theorems for a class of functional spaces *Izv. Math.* **70** 1051–76 (*Preprint* 0708.0806)

[22] Gracia-Bondia J M and Várilly J C 1988 Algebras of distributions suitable for phase-space quantum mechanics: I *J. Math. Phys.* **29** 869–79

[23] Soloviev M A 1997 An extension of distribution theory and of the Paley–Wiener–Schwartz theorem related to quantum gauge theory *Commun. Math. Phys.* **184** 579–96 (*Preprint* hep-th/9601005)

[24] Soloviev M A 1982 Space-like asymptotic behavior of vacuum expectation values in nonlocal field theory *Theor. Math. Phys.* **52** 854–62

[25] Smirnov A G 2003 Towards Euclidean theory of infrared singular quantum fields *J. Math. Phys.* **44** 2058–76 (*Preprint* math-ph/0205024)

[26] Chaichian M, Nishijima K and Tureanu A 2003 Spin-statistics and CPT-theorems in noncommutative field theories *Phys. Lett.* B **568** 146–52 (*Preprint* hep-th/0209008)

[27] Greenberg O W 2006 Failure of microcausality in quantum field theory on noncommutative spacetime *Phys. Rev.* D **73** 045014 (*Preprint* hep-th/0508057)

[28] Schroer B 2005 An antology of non-local QFT and QFT on noncommutative spacetime *Ann. Phys.* **319** 92 (*Preprint* hep-th/0405105)

[29] Borchers H J and Pohlmeyer K 1968 Eine scheinbare Abschwächung der Lokalitäts-bedingung: II *Commun. Math. Phys.* **8** 269–81

[30] Smirnov A G and Soloviev M A 2001 On Wick power series convergent to nonlocal fields *Theor. Math. Phys.* **127** 632–45 (*Preprint* math-ph/0104007)

[31] Chaichian M, Kulish P P, Nshijima K and Tureanu A 2004 On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT *Phys. Lett.* B **604** 98–102 (*Preprint* hep-th/0508057)